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An analytic approach to the unsteady heat conduction processes in one-dimensional composite media

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Abstract

The transient heat conduction problems in one-dimensional multi-layer solids are usually solved applying conventional techniques based on Vodicka's approach. However, if the thermal diffusivity of each layer is retained on the side of the heat conduction equation modified from the application of the separation-of-variables method where the time-dependent function is collected, then the modified heat conduction equation by itself represents a transparent statement of the physical phenomena involved. This 'natural' choice so simplifies unsteady heat conduction analysis of composite media that thermal response computation reduces to a matter of relatively simple mathematics when compared with traditional techniques heretofore employed. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

An exact closed-form solution for the transient temperature response of multi-layer series composite solids was originally obtained by Vodicka [1]. He determined the solution applying the method of separation of variables to the heat conduction partial differential equation. In separating the variables, Vodicka retained the thermal diffusivity on the side of the modified heat conduction equation where the space-variable function is collected [1,2]. This choice makes the time-variable function which appears in the series solution of the problem explicitly independent of the thermal diffusivity. For this reason the solution does not represent the physical reality of the problem, although it yields correct quantitative results, and the computation of the eigenvalues and corresponding eigenfunctions results is a quite lengthy and difficult matter.

After Vodicka, the analysis of unsteady-state heat conduction in composite solids has been under development for some 50 years and includes individual contributions of inspired quality. Noteworthy attempts are those of Mikhailov et al. [3], Carslaw and Jaeger [4], Huang and Chang [5] and Feng and Michaelides [6], Haji-Sheikh and Beck [7], and Yener and Özişik [8], to which correspond, respectively, the orthogonal expansion technique, the Laplace transform method, the Green's function approach, the Galerkin procedure and the finite integral transform technique. The reader may refer to Özişik [9] for a quite complete review of the specialized literature. However, with the exception of Carslaw and Jaeger [4], who only considered regions of infinite and semi-infinite thickness, all attempts drew on Vodicka' approach. Herein lies their mathematical difficulty, which is mainly related to the determination of the eigenvalues and corresponding eigenfunctions. In fact, in transient heat conduction the thermal diffusivity acts straight on only the time-dependent function, and the method of Vodicka is ill equipped to do it. Concerning this, Nietzsche would say that a difficulty persists only for as long as it is subject to inappropriate (although outstanding) methods of attack [10].

As a matter of fact, another exact closed-form solution for the transient thermal response of multi-layer composite media was independently derived by Tittle [11]. He applied the separation-of-variables method to the problem at issue and, with an appropriate choice concerning the position of the thermal diffusivity in the

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Nomenclature

Bi_1, Bi_{M+1}	Biot numbers at the outer boundary surfaces: h_1x_1/k_1 and $h_{M+1}x_1/k_1$
c	integration coefficient
$f_i(x)$	arbitrary initial temperature distribution for the ith layer
$F_i(x)$	arbitrary initial temperature difference for the ith layer: $T_\infty - f_i(x)$
h_1, h_{M+1}	convective heat transfer coefficients at the outer boundary surfaces
J_0	zero-order Bessel function of the first kind
J_1	first-order Bessel function of the first kind
k_i	thermal conductivity of the ith layer
M	number of layers
N	norm
q	integer number ($q = 0$, 1, 2 for plate, cylinder and sphere, respectively)
t	time
T_i	temperature for the ith layer of the composite medium
T_0	uniform initial temperature of the composite domain
T_∞	fluid temperature
x	space coordinate (either rectangular, cylindrical or spherical)
x_1, x_{M+1}	values of the space coordinate at the outer boundary surfaces
x_i	values of the space coordinate at the inner boundary surfaces ($i = 2, 3, \ldots, M$)
$\tilde{X}_i$	solutions to Eqs. (10) ($i = 1, 2, \ldots, M$)
$\tilde{X}_{i,m}$	mth eigenfunction corresponding to β_m for the ith layer
Y_0	zero-order Bessel function of the second kind
Y_1	first-order Bessel function of the second kind

Greek symbols

α_i	thermal diffusivity of the ith layer
β	dimensionless constant of separation: λx_1
β_m	mth dimensionless eigenvalue: $\lambda_m x_1$
γ_i	geometric ratio: x_i/x_1 ($i = 1, 2, \ldots, M, M+1$)
δ_i	thermal diffusivity ratio: α_i/α_1 ($i = 1, 2, \ldots, M$)
θ_i	temperature difference for the ith layer: $T_\infty - T_i$
θ_0	uniform initial temperature difference: $T_\infty - T_0$
Θ_i	dimensionless temperature for the ith layer: θ_i/θ_0
κ_i	thermal conductivity ratio: k_i/k_1 ($i = 1, 2, \ldots, M$)
λ	constant of separation
λ_m	mth eigenvalue
ξ	dimensionless space coordinate: x/x_1
Π_i	functions defined by Eqs. (18a)–(18c) and (22a)–(22c) (for three-layer cylinder see Eqs. (42a)–(42c))
τ	dimensionless time (or Fourier number): $\alpha_1 t/x_1^2$
Φ_i	functions defined by Eqs. (15) (for three-layer cylinder see Eqs. (43))
$\Phi_{i,i-1}$	functions defined by Eqs. (17a), (17b) and (21a), (21b)

Subscripts

i	ith layer defined in the domain $x_i \leqslant x \leqslant x_{i+1}$ ($i = 1, 2, \ldots, M$)
m	integer number (positive)
n	integer number (positive)

Superscripts

$+$	dimensionless
$'$	first derivative with respect to x

modified heat conduction equation, he made the variable–time function dependent on this thermal property. Then, Tittle first established physical constraints expressed by means of mathematical relationships linking the eigenvalues related to each of the M layers which are usually different in the different regions of the composite solid [11]. These relationships are the *essence* of composite medium analysis itself. They follow straight from the continuity of both temperature and conduction heat flux at the surfaces of separation of the composite body when perfect thermal contact between adjacent layers is verified. However, the calculation of the quasi-orthogonal eigenfunctions corresponding to the eigenvalues of the problem is a quite lengthy and difficult matter. The specialized literature indicates only one transient multilayer work [12] which drew on Tittle's source.

A 'natural' analytic approach for solving one-dimensional transient heat conduction in a two-layer series composite slab was recently developed by de Monte [13]. It combines the efficiency of Tittle's approach for the determination of the physical constraints relating the eigenvalues with the simplicity of Vodicka's approach for the calculation of the orthogonality relation linking the eigenfunctions.

In the present paper the proposed method is applied to composites of any number of layers, and in particular it allows composite media of rectangular, cylindrical and spherical layers which are in perfect thermal contact to be simultaneously treated. The combined method is relatively simple and particularly convenient and powerful when compared with classical methods so far employed. In fact, it yields the transcendental equation (eigencondition) for the determination of the eigenvalues in a definitive form which results to be virtually and symbolically independent of the number of layers. Also the algebraic expressions for the calculation of the coefficients which appear in the corresponding eigenfunctions are given in a definitive form for each of the M layers. Moreover, a new type of orthogonality relationship is also developed by the author and then used as a straightforward matter to achieve the final series form of the exact closed-form solution.

The treatment that follows presents the essence of the 'natural' analytic approach itself, and shows that the unsteady heat conduction processes in composite solids may be solved without the problems usually associated with the conventional techniques based on Vodicka's source.

2. Mathematical modeling of M-layer unsteady heat conduction

Consider a composite solid consisting of M parallel layers in perfect thermal contact, as shown in Fig. 1. Let k_i and α_i be the thermal conductivity and the thermal diffusivity of the ith layer, respectively $(i = 1, 2, \ldots, M)$. Initially $(t = 0)$ the body, which is confined to the domain $x_1 \leqslant x \leqslant x_{M+1}$, is at a specified temperature $f(x)$. Suddenly, at $t = 0$, both boundary surfaces of the composite solid are subjected to convection heat flux. In particular, a fluid at a temperature T_∞ with a heat transfer coefficient h_1 flows over the outer surface $x = x_1$, and another fluid at the same temperature T_∞ but with a different heat transfer coefficient h_{M+1} flows over the other outer surface $x = x_{M+1}$.

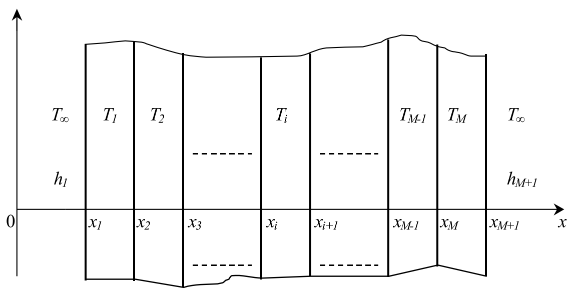

Fig. 1. Schematic representation of a multi-layer composite medium.

The assumptions made in deriving the mathematical modeling of the proposed unsteady heat conduction process are: (a) there is no heat generation within the body; (b) the thermal properties, i.e., conductivity and diffusivity, are independent of temperature and are uniform within each of the M layers; (c) the temperature T_∞ of the fluid surrounding the medium is spatially uniform and maintained constant for times $t > 0$; (d) the multi-layer solid is sufficiently large in the y and z directions in comparison to its thickness in the x direction; and (e) the heat transfer coefficients h_1 and h_{M+1} are uniform and constant.

Therefore, the heat conduction problem at issue can be considered *linear*, *one-dimensional*, and also *homogeneous* setting $\theta_i(x,t) = T_\infty - T_i(x,t)$ $(i = 1, 2, \ldots M)$ [9]. Its final mathematical formulation in a generic coordinate system, namely either rectangular, cylindrical or spherical, may be given as $(t \geqslant 0)$

- Heat conduction differential equations:

$$\frac{1}{x^q}\frac{\partial}{\partial x}\left(x^q\frac{\partial\theta_i}{\partial x}\right) = \frac{1}{\alpha_i}\frac{\partial\theta_i}{\partial t}, \quad x \in [x_i, x_{i+1}] \quad (i = 1, 2, \ldots, M), \tag{1}$$

where $q = 0$, 1, 2 for plate, cylinder and sphere, respectively.

- Outer boundary condition $(x = x_1)$:

$$-k_1\left(\frac{\partial\theta_1}{\partial x}\right)_{x_1} + h_1\theta_1(x_1, t) = 0. \tag{2}$$

- Inner boundary conditions $(x = x_i)$:

$$\theta_{i-1}(x_i, t) = \theta_i(x_i, t) \quad (i = 2, 3, \ldots, M), \tag{3}$$

$$k_{i-1}\left(\frac{\partial\theta_{i-1}}{\partial x}\right)_{x_i} = k_i\left(\frac{\partial\theta_i}{\partial x}\right)_{x_i} \quad (i = 2, 3, \ldots, M). \tag{4}$$

- Outer boundary condition $(x = x_{M+1})$:

$$k_M\left(\frac{\partial\theta_M}{\partial x}\right)_{x_{M+1}} + h_{M+1}\theta_M(x_{M+1}, t) = 0. \tag{5}$$

- Initial conditions:

$$\theta_i(x, t = 0) = F_i(x), \quad x \in [x_i, x_{i+1}] \quad (i = 1, 2, \ldots, M). \tag{6}$$

Eqs. (3) say that at the surfaces of separation x_i $(i = 2, 3, \ldots M)$ of the M-layer solid the temperatures in two adjacent regions are the same, which will be valid for only perfect thermal contact. Eqs. (4), instead, state that the conduction heat flux is continuous in correspondence to each inner boundary surface [14]. The set of equations (1)–(6) can analytically be solved as shown in the next section.

3. A '*natural*' analytic technique for solving M-layer unsteady heat conduction

Eqs. (1) may be solved by assuming product solutions (*separation-of-variables method*) defined as [9]

$$\theta_i(x,t) = X_i(x)G_i(t), \quad t \geqslant 0, \; x \in [x_i, x_{i+1}] \; (i = 1,2,\ldots,M). \tag{7}$$

Substituting Eqs. (7) in Eqs. (1), we obtain the modified heat conduction equations:

$$\frac{1}{x^q}\frac{1}{X_i}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^q\frac{\mathrm{d}X_i}{\mathrm{d}x}\right) = \frac{1}{\alpha_i G_i}\frac{\mathrm{d}G_i}{\mathrm{d}t} = -\lambda_i^2, \quad t \geqslant 0, \; x \in [x_i, x_{i+1}] \; (i = 1,2,\ldots,M), \tag{8}$$

where λ_i $(i = 1,2,\ldots M)$ are the separation constants, each corresponding to its own layer. It will be shown afterwards that physical constraints link these constants. In separating the variables, the thermal diffusivities α_i have been retained on the left-hand side of Eqs. (8) where the time-dependent functions are collected. This '*natural*' way we proceed makes the functions $G_i(t)$ explicitly dependent on the corresponding thermal diffusivities, and consequently the analytic solution to the problem consistent with the physical reality of the transient heat conduction processes. The '*natural*' separation given by Eqs. (8) produces 2 by M ordinary differential equations

$$\frac{\mathrm{d}G_i}{\mathrm{d}t} + \lambda_i^2\alpha_i G_i = 0, \quad t \geqslant 0 \; (i = 1,2,\ldots,M), \tag{9}$$

$$\frac{1}{x^q}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^q\frac{\mathrm{d}X_i}{\mathrm{d}x}\right) + \lambda_i^2 X_i = 0, \quad x \in [x_i, x_{i+1}] \; (i = 1,2,\ldots,M). \tag{10}$$

The solutions for the time-variable functions are readily obtained from Eqs. (9) as

$$G_i(t) = \mathrm{e}^{-\lambda_i^2\alpha_i t}, \quad t \geqslant 0 \; (i = 1,2,\ldots,M). \tag{11}$$

The solutions for the space-dependent functions, instead, are determined by solving *Helmholtz equations* (10), which are characterized by only one space-coordinate, x. They can be taken as

$$X_i(x) = a_i X_a(\lambda_i, x) + b_i X_b(\lambda_i, x), \quad x \in [x_i, x_{i+1}] \; (i = 1,2,\ldots,M), \tag{12}$$

where $X_a(\lambda_i, x)$ and $X_b(\lambda_i, x)$ are linearly independent solutions of Eqs. (10), and a_i and b_i are the integration constants related to the ith layer of the composite medium. Table 1 shows the functions $X_a(\lambda_i, x)$ and $X_b(\lambda_i, x)$ for composites of rectangular, cylindrical and spherical layers.

Table 1
Linearly independent solutions $X_a(\lambda_i, x)$ and $X_b(\lambda_i, x)$ of Eqs. (10) for slabs, cylinders and spheres $(i = 1,2,\ldots,M)$

q (geometry)	$X_a(\lambda_i, x)$	$X_b(\lambda_i, x)$
0 (slab)	$\sin(\lambda_i x)$	$\cos(\lambda_i x)$
1 (cylinder)	$J_0(\lambda_i x)$	$Y_0(\lambda_i x)$
2 (sphere)	$\dfrac{\sin(\lambda_i x)}{x}$	$\dfrac{\cos(\lambda_i x)}{x}$

3.1. Application of boundary conditions

By requiring that the solutions $\theta_i(x,t) = X_i(x)G_i(t)$ $(i = 1,2,\ldots,M)$ satisfy the boundary conditions (2)–(5), the following results are derived:

$$X_i(x) = a_1\Phi_i(\lambda_1,\ldots,\lambda_i)\tilde{X}_i(\lambda_1,\ldots,\lambda_i,x), \quad x \in [x_i, x_{i+1}] \; (i = 1,2,\ldots,M), \tag{13}$$

$$\lambda_i = \lambda_1\sqrt{\alpha_1/\alpha_i} \quad (i = 2,3,\ldots,M). \tag{14}$$

Eqs. (14) constrain the constants of separation λ_i and the thermal diffusivities α_i, which are usually different in the different regions of the composite medium. They are *fundamental* in order to ensure that the inner boundary conditions (3) and (4) are verified, since the thermal diffusivity is in general discontinuous at the surfaces of separation of the layers. As a matter of fact, these relationships were originally derived by Tittle [11].

The functions Φ_i and $\tilde{X}_i$ $(i = 1,2,\ldots,M)$, which appear in Eqs. (13), are given by

$$\Phi_1(\lambda_1) = 1; \quad \Phi_i(\lambda_1,\ldots,\lambda_i) = \Phi_{i,i-1}\Phi_{i-1,i-2}\cdots\Phi_{3,2}\Phi_{2,1} \quad (i = 2,3,\ldots,M), \tag{15}$$

$$\tilde{X}_i(\lambda_1,\ldots,\lambda_i,x) = X_a(\lambda_i,x) + \Pi_i(\lambda_1,\ldots,\lambda_i)X_b(\lambda_i,x), \quad x \in [x_i, x_{i+1}] \; (i = 1,2,\ldots,M). \tag{16}$$

The functions $\Phi_{i,i-1}$ $(i = 2,3,\ldots,M)$ appearing in Eqs. (15) are defined as

$$\Phi_{i,i-1}(\lambda_1,\ldots,\lambda_i) = \frac{\tilde{X}_{i-1}(\lambda_1,\ldots,\lambda_{i-1},x_i)}{\tilde{X}_i(\lambda_1,\ldots,\lambda_i,x_i)} \quad (i = 2,3,\ldots,M), \tag{17a}$$

$$\Phi_{M,M-1}(\lambda_1,\ldots,\lambda_M) = \frac{k_{M-1}}{k_M}\frac{\tilde{X}'_{M-1}(\lambda_1,\ldots,\lambda_{M-1},x_M)}{\tilde{X}'_M(\lambda_1,\ldots,\lambda_M,x_M)}. \tag{17b}$$

It may be noted that the function $\Phi_{M,M-1}$ can be obtained both by means of Eq. (17a) setting $i = M$ and by means of Eq. (17b). Instead the functions Π_i $(i = 1,2,\ldots,M)$, which explicitly appear in Eqs. (16) and implicitly appear in Eqs. (17a) and (17b) by means of $\tilde{X}_i$, are given by

$$\Pi_1(\lambda_1) = -\frac{h_1 X_a(\lambda_1, x_1) - k_1 X_a'(\lambda_1, x_1)}{h_1 X_b(\lambda_1, x_1) - k_1 X_b'(\lambda_1, x_1)}, \tag{18a}$$

$$\begin{aligned}\Pi_i(\lambda_1, \ldots, \lambda_i) = &-\left[k_i X_a'(\lambda_i, x_i)\tilde{X}_{i-1}(\lambda_1, \ldots, \lambda_{i-1}, x_i)\right.\\ &\left.- k_{i-1} X_a(\lambda_i, x_i)\tilde{X}_{i-1}'(\lambda_1, \ldots, \lambda_{i-1}, x_i)\right]\\ &\Big/\left[k_i X_b'(\lambda_i, x_i)\tilde{X}_{i-1}(\lambda_1, \ldots, \lambda_{i-1}, x_i)\right.\\ &\left.- k_{i-1} X_b(\lambda_i, x_i)\tilde{X}_{i-1}'(\lambda_1, \ldots, \lambda_{i-1}, x_i)\right]\\ &(i = 2, 3, \ldots, M),\end{aligned} \tag{18b}$$

$$\Pi_M(\lambda_M) = -\frac{h_{M+1} X_a(\lambda_M, x_{M+1}) + k_M X_a'(\lambda_M, x_{M+1})}{h_{M+1} X_b(\lambda_M, x_{M+1}) + k_M X_b'(\lambda_M, x_{M+1})}. \tag{18c}$$

It may still be observed that the function Π_M can be evaluated by both Eq. (18b) for $i = M$ and Eq. (18c). In view of Eqs. (14), these two equations depend on only the constant of separation λ_1 (see next section).

3.2. Application of Eqs. (14)

Bearing in mind the $M - 1$ relationships (14) and setting $\lambda_1 = \lambda$ and $a_1 = c$, the solutions $\theta_i(x, t)$ $(i = 1, 2, \ldots, M)$ assume the following expressions:

$$\begin{aligned}&\theta_i(x, t) = c\Phi_i(\lambda)\tilde{X}_i(\lambda, x)\,\mathrm{e}^{-\lambda^2\alpha_1 t}, \quad t \geqslant 0,\\ &x \in [x_i, x_{i+1}] \ (i = 1, 2, \ldots, M),\end{aligned} \tag{19}$$

where the functions $\tilde{X}_i$ $(i = 1, 2, \ldots, M)$ become

$$\begin{aligned}&\tilde{X}_i(\lambda, x) = X_a(\lambda\sqrt{\alpha_1/\alpha_i}, x) + \Pi_i(\lambda) X_b(\lambda\sqrt{\alpha_1/\alpha_i}, x),\\ &x \in [x_i, x_{i+1}] \ (i = 1, 2, \ldots, M).\end{aligned} \tag{20}$$

The linearly independent solutions of Eqs. (10), i.e., $X_a(\lambda\sqrt{\alpha_1/\alpha_i}, x)$ and $X_b(\lambda\sqrt{\alpha_1/\alpha_i}, x)$, which appear in Eqs. (20), may be obtained from Table 1 simply setting $\lambda_i = \lambda\sqrt{\alpha_1/\alpha_i}$ $(i = 1, 2, \ldots, M)$. Additionally, the functions Φ_i $(i = 1, 2, \ldots, M)$ are still defined by means of Eqs. (15), while the inherent functions $\Phi_{i,i-1}$ $(i = 2, 3, \ldots, M)$ may be rewritten as

$$\Phi_{i,i-1}(\lambda) = \frac{\tilde{X}_{i-1}(\lambda, x_i)}{\tilde{X}_i(\lambda, x_i)} \quad (i = 2, 3, \ldots, M), \tag{21a}$$

$$\Phi_{M,M-1}(\lambda) = \frac{k_{M-1}}{k_M}\frac{\tilde{X}_{M-1}'(\lambda, x_M)}{\tilde{X}_M'(\lambda, x_M)}. \tag{21b}$$

As far as the functions Π_i $(i = 1, 2, \ldots, M)$ are concerned, they become

$$\Pi_1(\lambda) = -\frac{h_1 X_a(\lambda, x_1) - k_1 X_a'(\lambda, x_1)}{h_1 X_b(\lambda, x_1) - k_1 X_b'(\lambda, x_1)}, \tag{22a}$$

$$\begin{aligned}\Pi_i(\lambda) = &-\left[k_i X_a'(\lambda\sqrt{\alpha_1/\alpha_i}, x_i)\tilde{X}_{i-1}(\lambda, x_i)\right.\\ &\left.- k_{i-1} X_a(\lambda\sqrt{\alpha_1/\alpha_i}, x_i)\tilde{X}_{i-1}'(\lambda, x_i)\right]\\ &\Big/\left[k_i X_b'(\lambda\sqrt{\alpha_1/\alpha_i}, x_i)\tilde{X}_{i-1}(\lambda, x_i)\right.\\ &\left.- k_{i-1} X_b(\lambda\sqrt{\alpha_1/\alpha_i}, x_i)\tilde{X}_{i-1}'(\lambda, x_i)\right]\\ &(i = 2, 3, \ldots, M),\end{aligned} \tag{22b}$$

$$\begin{aligned}\Pi_M(\lambda) = &-\left[h_{M+1} X_a(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1})\right.\\ &\left.+ k_M X_a'(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1})\right]\\ &\Big/\left[h_{M+1} X_b(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1})\right.\\ &\left.+ k_M X_b'(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1})\right].\end{aligned} \tag{22c}$$

Comparing Eq. (21a) for $i = M$ and Eq. (21b), as well as Eq. (22b) for $i = M$ and Eq. (22c), yields the following set of algebraic equations:

$$\frac{\tilde{X}_{M-1}(\lambda, x_M)}{\tilde{X}_M(\lambda, x_M)} - \frac{k_{M-1}}{k_M}\frac{\tilde{X}_{M-1}'(\lambda, x_M)}{\tilde{X}_M'(\lambda, x_M)} = 0, \tag{23}$$

$$\begin{aligned}&\left[k_M X_a'(\lambda\sqrt{\alpha_1/\alpha_M}, x_M)\tilde{X}_{M-1}(\lambda, x_M)\right.\\ &\left.- k_{M-1} X_a(\lambda\sqrt{\alpha_1/\alpha_M}, x_M)\tilde{X}_{M-1}'(\lambda, x_M)\right]\\ &\Big/\left[k_M X_b'(\lambda\sqrt{\alpha_1/\alpha_M}, x_M)\tilde{X}_{M-1}(\lambda, x_M)\right.\\ &\left.- k_{M-1} X_b(\lambda\sqrt{\alpha_1/\alpha_M}, x_M)\tilde{X}_{M-1}'(\lambda, x_M)\right]\\ &- \left[h_{M+1} X_a(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1}) + k_M X_a'(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1})\right]\\ &\Big/\left[h_{M+1} X_b(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1}) + k_M X_b'(\lambda\sqrt{\alpha_1/\alpha_M}, x_{M+1})\right] = 0.\end{aligned} \tag{24}$$

Hence Eqs. (19) simultaneously satisfy (1)–(5), where c is a constant depending on the initial conditions (6) and λ is any number other than zero (in particular, greater than zero for cylinders) which simultaneously verifies the transcendental equations (23) and (24). However, only Eq. (24) represents the *eigencondition* of the considered unsteady-state heat conduction problem and allows the corresponding *eigenvalues* λ_m $(m = 1, 2, 3, \ldots)$ to be calculated. In fact, substituting Eq. (20) for $i = M$ in Eq. (23), it can be proven with appropriate manipulations that this equation reduces to only Eq. (22b) for $i = M$. Consequently, Eq. (23) does not give any useful information concerning the eigenvalues of the problem.

Therefore, there are numerous solutions having the forms (19), each corresponding to a consecutive value of the eigenvalues $\lambda_1 < \lambda_2 < \cdots \lambda_m < \cdots$ $(m = 1, 2, 3, \ldots)$:

$$\begin{aligned}&\theta_{i,m}(x, t) = c_m \Phi_{i,m}\tilde{X}_{i,m}(x)\,\mathrm{e}^{-\lambda_m^2\alpha_1 t},\\ &t \geqslant 0, \ x \in [x_i, x_{i+1}] \ (i = 1, 2, \ldots, M),\end{aligned} \tag{25}$$

where $\Phi_{i,m} = \Phi_i(\lambda_m)$; of course, $\Phi_{i,i-1,m} = \Phi_{i,i-1}(\lambda_m)$. The functions $\tilde{X}_{i,m}(x) = \tilde{X}_i(\lambda_m, x)$ $(i = 1, 2, \ldots, M)$ are the *eigenfunctions* corresponding to the eigenvalues λ_m, and are defined as

$$\tilde{X}_{i,m}(x) = X_a(\lambda_m\sqrt{\alpha_1/\alpha_i}, x) + \Pi_{i,m}X_b(\lambda_m\sqrt{\alpha_1/\alpha_i}, x),\quad x \in [x_i, x_{i+1}]\ (i = 1, 2, \ldots, M), \tag{26}$$

where $\Pi_{i,m} = \Pi_i(\lambda_m)$. It may be proven (Appendix A) that the eigenfunctions $\tilde{X}_{i,m}(x)$ $(i = 1, 2, \ldots, M)$ defined before are orthogonal functions. In fact, they satisfy the new type of orthogonality relationship

$$\sum_{i=1}^{M}\Phi_{i,m}\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q\tilde{X}_{i,m}(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x = \begin{cases} 0 & \text{for } m \neq n, \\ N_m & \text{for } m = n, \end{cases} \tag{27}$$

which may be called '*natural*' orthogonality property according to the '*natural*' choice performed in separating the variables, as shown in the modified heat conduction equations (8). The functions $\tilde{X}_{i,m}(x)$ and $\tilde{X}_{i,n}(x)$ appearing in Eqs. (27) represent two different eigenfunctions for the ith layer corresponding to two different eigenvalues λ_m and λ_n, respectively. The constant N_m, called normalization integral (norm) and connected to λ_m, is defined as

$$N_m = \sum_{i=1}^{M}(\Phi_{i,m})^2\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q(\tilde{X}_{i,m})^2\,\mathrm{d}x, \tag{28}$$

and, for this particular case, its inherent integrals may be evaluated as (Appendix B)

$$\int_{x_i}^{x_{i+1}} x^q(\tilde{X}_{i,m})^2\,\mathrm{d}x = \left[\frac{x^{q+1}(\tilde{X}_{i,m})^2}{2}\right]_{x_i}^{x_{i+1}} + \left[\frac{x^{q+1}(\tilde{X}'_{i,m})^2}{2\lambda_m^2(\alpha_1/\alpha_i)}\right]_{x_i}^{x_{i+1}} + (q-1)\left[\frac{x^q\tilde{X}_{i,m}\tilde{X}'_{i,m}}{2\lambda_m^2(\alpha_1/\alpha_i)}\right]_{x_i}^{x_{i+1}}\quad (i = 1, 2, \ldots, M), \tag{29}$$

where $q = 0$, 1, 2 for plate, cylinder and sphere, respectively. It may be noted that the last term on the right-hand side of Eqs. (29) vanishes for composites of cylindrical parallel layers ($q = 1$).

Then the complete solutions for the temperature distributions $\theta_i(x, t)$ $(i = 1, 2, \ldots, M)$ are constructed by taking a linear sum of all individual solutions given by Eqs. (25):

$$\theta_i(x, t) = \sum_{m=1}^{\infty} c_m\Phi_{i,m}\tilde{X}_{i,m}(x)\,\mathrm{e}^{-\lambda_m^2\alpha_1 t},\quad t \geqslant 0,\ x \in [x_i, x_{i+1}]\ (i = 1, 2, \ldots, M), \tag{30}$$

where the summation is over all eigenvalues. The final series forms of the solutions satisfy the heat conduction partial differential equations (1), and the boundary conditions (2)–(5). But they do not satisfy the initial conditions (6), which can be applied to evaluate the last unknown coefficients c_m.

3.3. Application of initial conditions

We now constrain the solutions (30) to satisfy the corresponding initial conditions expressed by (6), and obtain

$$F_i(x) = \sum_{m=1}^{\infty} c_m\Phi_{i,m}\tilde{X}_{i,m}(x),\quad x \in [x_i, x_{i+1}]\ (i = 1, 2, \ldots, M). \tag{31}$$

The coefficients c_m can be determined by using the orthogonality relation (27) as now described. Both sides of Eqs. (31) can be multiplied by $x^q(k_i/\alpha_i)\Phi_{i,n}\tilde{X}_{i,n}(x)$, and the resulting expressions can be integrated with respect to x from $x = x_i$ to $x = x_{i+1}$. It will result in

$$\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^qF_i(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x = \sum_{m=1}^{\infty} c_m\Phi_{i,m}\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q\tilde{X}_{i,m}(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x\quad (i = 1, 2, \ldots, M). \tag{32}$$

Summing up the resulting expressions (32) from $i = 1$ to M (i.e., over all domains of the composite medium), we have

$$\sum_{i=1}^{M}\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^qF_i(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x = \sum_{m=1}^{\infty} c_m\left[\sum_{i=1}^{M}\Phi_{i,m}\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q\tilde{X}_{i,m}(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x\right]. \tag{33}$$

In view of the orthogonality property (27), the term inside the square brackets on the right-hand side of Eq. (33) vanishes for $m \neq n$, and becomes equal to N_m expressed by Eqs. (28) and (29) for $m = n$. Therefore, the coefficients c_m are given by

$$c_m = \frac{1}{N_m}\sum_{i=1}^{M}\Phi_{i,m}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^qF_i(x)\tilde{X}_{i,m}(x)\,\mathrm{d}x. \tag{34}$$

The substitution of c_m from Eq. (34) in Eqs. (30) gives the final series forms of the solutions for the calculation of the thermal fields within each of the M layers of the composite solid. If the solid is initially at a uniform temperature T_0, then $F_i(x) = \theta_0$ $(i = 1, 2, \ldots, M)$, and the coefficients c_m are determined as

$$c_m = \frac{\theta_0}{N_m}\sum_{i=1}^{M}\Phi_{i,m}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q\tilde{X}_{i,m}(x)\,\mathrm{d}x, \tag{35}$$

where the integral appearing on the right-hand side of the previous equation may readily be solved. In fact, in view of Eq. (A.3) with $k = m$ and applying the integration by parts, we derive

$$\int_{x_i}^{x_{i+1}} x^q \tilde{X}_{i,m}(x)\,\mathrm{d}x = -\left[\frac{x^q \tilde{X}'_{i,m}(x)}{\lambda_m^2(\alpha_1/\alpha_i)}\right]_{x_i}^{x_{i+1}} \quad (i = 1,2,\ldots,M). \tag{36}$$

3.4. Dimensionless groups

From Eqs. (30), with the coefficients c_m determined by means of Eqs. (35) and (36), it follows that the temperatures $\theta_i(x,t)$ $(i = 1,2,\ldots,M)$ depend on the following physical parameters $(6 + 3M)$:

$$x,\ t,\ k_i,\ x_i,\ x_{M+1},\ h_1,\ h_{M+1},\ \alpha_i,\ \theta_0 \quad (i = 1,2,\ldots,M). \tag{37}$$

Buckingham's 'pi' theorem [15] states that, with four basic dimensions, mass [M], length [L], time [t] and temperature [T], which are sufficient for describing unsteady heat conduction by physical variables, a reduction of up to four may be hoped for in the number of these variables. Therefore, the process of applying the 'pi' theorem will result in a total of $2 + 4M$ dimensionless groups. There is a choice of algebraic techniques for determining a consistent set of groupings [14,15]. Applying any preferred technique results in the following set of groups (or in an equivalent set achieved by combination):

$$\xi,\ \tau,\ \kappa_i,\ \gamma_i,\ \gamma_{M+1},\ Bi_1,\ Bi_{M+1},\ \delta_i,\ \Theta_1,\ \Theta_i \quad (i = 2,3,\ldots,M). \tag{38}$$

It may be observed that the dimensionless groups (38), with the exception of Θ_i $(i = 1,2,\ldots,M)$, have been based on the first layer, as done in [16,17]. The nondimensional parameters listed before may be varied *in form* by algebraic combination, but the total number of *independent* groups $(2 + 4M)$ remains the same.

4. Application of the 'natural' analytic technique

In this section we illustrate the application of the '*natural*' analytic approach described in the previous section for the solution of the homogeneous problem of one-dimensional transient heat conduction in a composite medium consisting of three parallel cylindrical layers initially at a uniform temperature θ_0.

Introducing the normalized variables derived in Section 3.4, the dimensionless thermal fields Θ_i $(i = 1,2,3)$ for the problem at issue can be represented as follows:

$$\Theta_i(\xi,\tau) = \sum_{m=1}^{\infty} c_m^+ \Phi_{i,m} \tilde{X}_{i,m}(\xi)\,\mathrm{e}^{-\beta_m^2\tau}, \quad \tau \geqslant 0,\ \ \xi \in [\gamma_i, \gamma_{i+1}]\ \ (i = 1,2,3), \tag{39}$$

where $\gamma_1 = 1$. The quantity $\beta_m = \lambda_m x_1$ represents the mth dimensionless eigenvalue (root) of the following eigencondition:

$$\begin{aligned}&\Big[(\kappa_3/\sqrt{\delta_3})J_1(\beta\gamma_3/\sqrt{\delta_3})\tilde{X}_2(\gamma_3) - (\kappa_2/\sqrt{\delta_2})J_0(\beta\gamma_3/\sqrt{\delta_3})\Lambda_2(\gamma_3)\Big]\\ &\Big/\Big[(\kappa_3/\sqrt{\delta_3})Y_1(\beta\gamma_3/\sqrt{\delta_3})\tilde{X}_2(\gamma_3) - (\kappa_2/\sqrt{\delta_2})Y_0(\beta\gamma_3/\sqrt{\delta_3})\Lambda_2(\gamma_3)\Big] - \Big[Bi_4 J_0(\beta\gamma_4/\sqrt{\delta_3})\\ &- (\beta\kappa_3/\sqrt{\delta_3})J_1(\beta\gamma_4/\sqrt{\delta_3})\Big]\Big/\Big[Bi_4 Y_0(\beta\gamma_4/\sqrt{\delta_3}) - (\beta\kappa_3/\sqrt{\delta_3})Y_1(\beta\gamma_4/\sqrt{\delta_3})\Big] = 0.\end{aligned} \tag{40}$$

The functions $\tilde{X}_2(\gamma_3)$ and $\Lambda_2(\gamma_3)$ appearing in Eq. (40) may be obtained from relations (41) and (45), respectively, simply setting $i = 2$, $\beta_m = \beta$ and $\xi = \gamma_3$. The eigenfunctions $\tilde{X}_{i,m}(\xi)$ $(i = 1,2,3)$ in the solutions (39) assume the following expressions:

$$\tilde{X}_{i,m}(\xi) = J_0(\beta_m\xi/\sqrt{\delta_i}) + \Pi_{i,m} Y_0(\beta_m\xi/\sqrt{\delta_i}), \quad \xi \in [\gamma_i, \gamma_{i+1}]\ \ (i = 1,2,3), \tag{41}$$

where $\delta_1 = 1$. In particular, the functions $\Pi_{i,m}$ $(i = 1,2,3)$ may be evaluated as

$$\Pi_{1,m} = -\frac{Bi_1 J_0(\beta_m) + \beta_m J_1(\beta_m)}{Bi_1 Y_0(\beta_m) + \beta_m Y_1(\beta_m)}, \tag{42a}$$

$$\begin{aligned}\Pi_{2,m} = -&\Big[(\kappa_2/\sqrt{\delta_2})J_1(\beta_m\gamma_2/\sqrt{\delta_2})\tilde{X}_{1,m}(\gamma_2) - J_0(\beta_m\gamma_2/\sqrt{\delta_2})\Lambda_{1,m}(\gamma_2)\Big]\\ &\Big/\Big[(\kappa_2/\sqrt{\delta_2})Y_1(\beta_m\gamma_2/\sqrt{\delta_2})\tilde{X}_{1,m}(\gamma_2) - Y_0(\beta_m\gamma_2/\sqrt{\delta_2})\Lambda_{1,m}(\gamma_2)\Big],\end{aligned} \tag{42b}$$

$$\begin{aligned}\Pi_{3,m} = -&\Big[Bi_4 J_0(\beta_m\gamma_4/\sqrt{\delta_3}) - (\beta_m\kappa_3/\sqrt{\delta_3})J_1(\beta_m\gamma_4/\sqrt{\delta_3})\Big]\Big/\Big[Bi_4 Y_0(\beta_m\gamma_4/\sqrt{\delta_3})\\ &- (\beta_m\kappa_3/\sqrt{\delta_3})Y_1(\beta_m\gamma_4/\sqrt{\delta_3})\Big],\end{aligned} \tag{42c}$$

where the quantities $\tilde{X}_{1,m}(\gamma_2)$ and $\Lambda_{1,m}(\gamma_2)$ can be evaluated by means of Eqs. (41) and (45), respectively, simply setting $i = 1$ and $\xi = \gamma_2$. Instead the functions $\Phi_{i,m}$ $(i = 1,2,3)$, which appear in the solutions (39), become

$$\Phi_{1,m} = 1; \quad \Phi_{2,m} = \frac{\tilde{X}_{1,m}(\gamma_2)}{\tilde{X}_{2,m}(\gamma_2)}; \quad \Phi_{3,m} = \frac{\tilde{X}_{1,m}(\gamma_2)\tilde{X}_{2,m}(\gamma_3)}{\tilde{X}_{2,m}(\gamma_2)\tilde{X}_{3,m}(\gamma_3)}, \tag{43}$$

where the values of the eigenfunctions $\tilde{X}_{1,m}(\gamma_2)$, $\tilde{X}_{2,m}(\gamma_2)$, $\tilde{X}_{2,m}(\gamma_3)$ and $\tilde{X}_{3,m}(\gamma_3)$ may readily be obtained by means of Eq. (41) simply setting i equal to either 1, 2 or 3 and ξ equal to either γ_2 or γ_3. In view of Eqs. (35) and (36) with $q=1$, the dimensionless coefficient $c_m^+ = c_m/\theta_0$ appearing in the thermal fields (39) may be calculated by the following expressions ($\kappa_1 = 1$):

$$c_m^+ = \frac{1}{N_m^+ \beta_m} \sum_{i=1}^{3} \Phi_{i,m} \left(\frac{\kappa_i}{\sqrt{\delta_i}} \right) [\xi \Lambda_{i,m}(\xi)]_{\gamma_i}^{\gamma_{i+1}}, \tag{44}$$

where N_m^+ represents the dimensionless norm N_m, defined as $N_m^+ = \alpha_1 N_m/(k_1 x_1^2)$, and the normalized functions $\Lambda_{i,m}(\xi)$ are given by

$$\Lambda_{i,m}(\xi) = J_1(\beta_m \xi/\sqrt{\delta_i}) + \Pi_{i,m} Y_1(\beta_m \xi/\sqrt{\delta_i}), \quad \xi \in [\gamma_i, \gamma_{i+1}] \ \ (i = 1,2,3). \tag{45}$$

When the dimensionless variables defined in Section 3.4 are used, the norm N_m expressed by Eqs. (28) and (29) with $q=1$ becomes

$$N_m^+ = \frac{1}{2} \sum_{i=1}^{3} (\Phi_{i,m})^2 \left(\frac{\kappa_i}{\delta_i} \right) \left\{ [\xi \tilde{X}_{i,m}(\xi)]^2 + [\xi \Lambda_{i,m}(\xi)]^2 \right\}_{\gamma_i}^{\gamma_{i+1}}. \tag{46}$$

4.1. Numerical results

With reference to the transient three-cylindrical-region problem here under consideration, we assume for the dimensionless quantities which appear in the relations of the previous section the following values:

$$\gamma_2 = 2, \quad \gamma_3 = 4, \quad \gamma_4 = 6, \quad \kappa_2 = 4, \quad \kappa_3 = 4, \quad \delta_2 = 4, \quad \delta_3 = 9, \quad Bi_1 = 1, \quad Bi_4 = 2. \tag{47}$$

Therefore, the transcendental equation (40) may numerically be solved for the determination of the eigenvalues of the problem. In spite of its algebraic complexity, it can easily be solved in view of the very high computing technology available today [18]. In particular, the number p of eigenvalues which has to be used in the series solutions Θ_i $(i = 1,2,3)$ defined by Eqs. (39) may be established by requiring that the exact ($p \to \infty$) and approximate (p = finite) solutions differ by not more than 3%, which is quite acceptable in most engineering applications. Of course, the maximum deviations between the exact and approximate non-dimensional temperatures are obtained for $\tau = 0$ in correspondence to the outer boundary surfaces. It can be proven that, when $p = 30$, the percentage deviation for $\tau = 0$ is less than 2.8% at $\xi = \gamma_4 = 6$ and less than 2% at $\xi = \gamma_1 = 1$. In particular, the first 30 eigenvalues of the eigencondition (40) are given in Table 2.

Fig. 2 shows the dimensionless thermal field for the considered three-layer composite cylinder both as a function of ξ with τ as a parameter (Fig. 2(a)) and versus τ for different values of the parameter ξ (Fig. 2(b)). As illustrated in Fig. 2(b), there exists a point of intersection when $\tau = 0.3$ between the curve "$\xi = 1$" and the curve "$\xi = 5$". Therefore, for $\tau > 0.3$ the heating ($\theta_0 > 0$) or cooling ($\theta_0 < 0$) process of the medium occurs more quickly at the inner cylindrical surface $\xi = 5$ than at the outer cylindrical surface $\xi = 1$ (the curve "$\xi = 5$" becomes steeper than the curve "$\xi = 1$"). This behavior can also be observed in Fig. 2(a) and it can be justified on the basis of the values given to the dimensionless variables of the problem, in particular $Bi_4 = 2$ against $Bi_1 = 1$ and $\delta_3 = 9$.

Table 2
First 30 roots (eigenvalues) of Eq. (40) when the values of the dimensionless variables are defined by (47)

m	β_m	Value of β_m
1	β_1	0.75250
2	β_2	1.83236
3	β_3	2.63508
4	β_4	3.79369
5	β_5	4.91960
6	β_6	6.05853
7	β_7	7.33670
8	β_8	8.22265
9	β_9	9.52359
10	β_{10}	10.8582
11	β_{11}	11.7651
12	β_{12}	13.0337
13	β_{13}	14.2109
14	β_{14}	15.3854
15	β_{15}	16.6749
16	β_{16}	17.5780
17	β_{17}	18.8996
18	β_{18}	20.2370
19	β_{19}	21.1472
20	β_{20}	22.4255
21	β_{21}	23.6065
22	β_{22}	24.7853
23	β_{23}	26.0748
24	β_{24}	26.9812
25	β_{25}	28.3078
26	β_{26}	29.6447
27	β_{27}	30.5554
28	β_{28}	31.8368
29	β_{29}	33.0186
30	β_{30}	34.1988

5. Conclusions

A 'natural' analytic approach for solving transient multi-layer problems has been derived. It can be applied to composite media of any number of layers in any frame of reference, i.e., either rectangular, cylindrical or spherical. A comparison with the traditional techniques

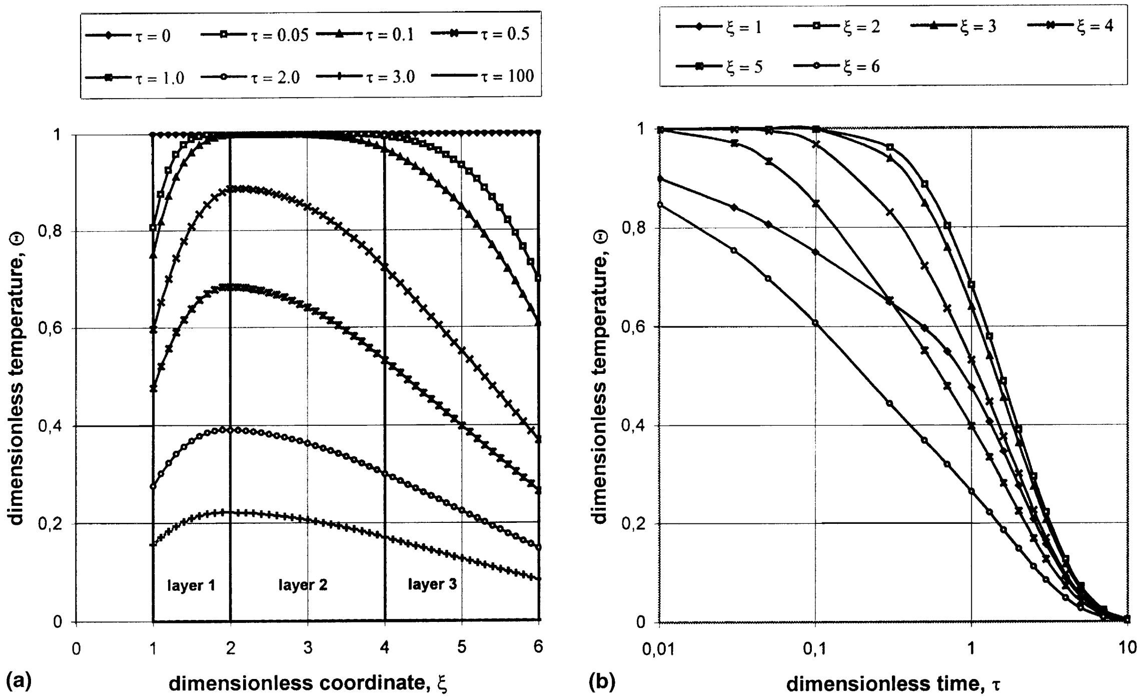

Fig. 2. Dimensionless temperature for a three-layer composite cylinder when the values of its dimensionless variables are defined by (47): (a) Θ vs. ξ with τ as a parameter; (b) Θ vs. τ with ξ as a parameter.

based on Vodicka's approach shows the following advantages of the proposed method:

- the coefficients a_i and b_i, which appear in the solutions of Helmholtz equations, are determined in a definitive form for each of the M layers with i as an index $(i = 1, 2, \ldots, M)$, and therefore their algebraic expressions can be applied to composites of any number of layers;
- the transcendental equation for the determination of the eigenvalues is obtained in a definitive form and results to be valid for composite media of any number of layers;
- the integrals inherent to the norm N_m are derived in a general form assuming the integer number q as a parameter, where $q = 0$, 1, 2 for composites of rectangular, cylindrical and spherical layers, respectively;
- the integrals inherent to the integration coefficient c_m are still determined in a general form with q as a parameter when the solid is initially at a uniform temperature.

A numerical example concerning a composite cylinder of three parallel layers initially at a uniform temperature has shown that the 'natural' analytic approach proposed by the author allows the calculation of the transient temperature within the considered body to be notably simplified.

Therefore, a large number of users might be able to handle the simplified task.

Appendix A

We shall now prove the '*natural*' orthogonality relation (27) for the eigenfunctions $\tilde{X}_{i,m}(x)$ $(i = 1, 2, \ldots, M)$ defined by Eqs. (26). The starting point is the calculation of the following expression:

$$\sum_{i=1}^{M} (k_i \Phi_{i,m} \Phi_{i,n} I_i), \tag{A.1}$$

where the integrals I_i are given by

$$I_i = \left(\frac{\alpha_1}{\alpha_i}\right)(\lambda_m^2 - \lambda_n^2) \int_{x_i}^{x_{i+1}} x^q \tilde{X}_{i,m}(x) \tilde{X}_{i,n}(x)\,\mathrm{d}x \quad (i = 1, 2, \ldots, M). \tag{A.2}$$

The eigenfunctions $\tilde{X}_{i,k}(x)$ $(i = 1, 2, \ldots, M$ and $k = m, n)$, which appear in the solutions (30) and are defined in the region $x_i \leqslant x \leqslant x_{i+1}$, satisfy the ordinary differential equations (10), where $\lambda_i^2 = \lambda_k^2(\alpha_1/\alpha_i)$ with $k = m$ and n, and the boundary conditions (2)–(5). Therefore, as the boundary conditions (3) are redundant for the proof here under consideration, it will result in

$$x^q \lambda_k^2 (\alpha_1/\alpha_i) \tilde{X}_{i,k} = -\frac{\mathrm{d}}{\mathrm{d}x}(x^q \tilde{X}'_{i,k}) \quad (i = 1, 2, \ldots, M) \ (k = m, n), \tag{A.3}$$

$$\tilde{X}'_{1,k}(x_1) = (h_1/k_1)\tilde{X}_{1,k}(x_1) \quad (k = m, n), \tag{A.4}$$

$$\tilde{X}'_{i,k}(x_i) = (k_{i-1}/k_i)\tilde{X}'_{i-1,k}(x_i)\Phi^{-1}_{i,i-1,k} \quad (i = 2, 3, \ldots, M) \;\; (k = m, n), \tag{A.5}$$

$$\tilde{X}'_{M,k}(x_{M+1}) = -(h_{M+1}/k_M)\tilde{X}_{M,k}(x_{M+1}) \quad (k = m, n). \tag{A.6}$$

Substituting Eqs. (A.3) in the integrals (A.2) and then applying the integration by parts, we get

$$I_i = x^q_{i+1} I_{i,i+1}(x_{i+1}) - x^q_i I_{i,i}(x_i) \quad (i = 1, 2, \ldots, M), \tag{A.7}$$

where

$$I_{i,i+1}(x_{i+1}) = \tilde{X}_{i,m}(x_{i+1})\tilde{X}'_{i,n}(x_{i+1}) - \tilde{X}_{i,n}(x_{i+1})\tilde{X}'_{i,m}(x_{i+1}) \quad (i = 1, 2, \ldots, M), \tag{A.8}$$

$$I_{i,i}(x_i) = \tilde{X}_{i,m}(x_i)\tilde{X}'_{i,n}(x_i) - \tilde{X}_{i,n}(x_i)\tilde{X}'_{i,m}(x_i) \quad (i = 1, 2, \ldots, M). \tag{A.9}$$

In view of Eqs. (A.4), expression (A.9) for $i = 1$ becomes

$$I_{1,1}(x_1) = 0. \tag{A.10}$$

Bearing in mind Eqs. (A.5), expressions (A.9) for $i = 2, 3, \ldots, M$ may be rewritten as

$$I_{i,i}(x_i) = (k_{i-1}/k_i)\Phi^{-1}_{i,i-1,m}\Phi^{-1}_{i,i-1,n} I_{i-1,i}(x_i) \quad (i = 2, 3, \ldots, M). \tag{A.11}$$

In view of Eqs. (A.6), expression (A.8) for $i = M$ becomes

$$I_{M,M+1}(x_{M+1}) = 0. \tag{A.12}$$

Bearing in mind Eqs. (A.10)–(A.12), integrals I_i $(i = 1, 2, \ldots, M)$ given by Eqs. (A.7) may be rewritten as

$$I_1 = x^q_2 I_{1,2}(x_2), \tag{A.13}$$

$$I_i = x^q_{i+1} I_{i,i+1}(x_{i+1}) - x^q_i (k_{i-1}/k_i)\Phi^{-1}_{i,i-1,m}\Phi^{-1}_{i,i-1,n} I_{i-1,i}(x_i) \quad (i = 2, 3, \ldots, M-1), \tag{A.14}$$

$$I_M = -x^q_M (k_{M-1}/k_M)\Phi^{-1}_{M,M-1,m}\Phi^{-1}_{M,M-1,n} I_{M-1,M}(x_M). \tag{A.15}$$

Substituting integrals (A.2) in the expression (A.1), we have

$$(\lambda^2_m - \lambda^2_n)\alpha_1 \sum_{i=1}^{M} \Phi_{i,m}\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q \tilde{X}_{i,m}(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x. \tag{A.16}$$

Substituting integrals (A.13)–(A.15) in the expression (A.1), we obtain that this expression vanishes. Therefore, the following result is derived:

$$(\lambda^2_m - \lambda^2_n)\sum_{i=1}^{M} \Phi_{i,m}\Phi_{i,n}\left(\frac{k_i}{\alpha_i}\right)\int_{x_i}^{x_{i+1}} x^q \tilde{X}_{i,m}(x)\tilde{X}_{i,n}(x)\,\mathrm{d}x = 0, \tag{A.17}$$

which immediately proves the orthogonality property (27).

Appendix B

The integrals inherent to the norm N_m defined by means of Eq. (28) can be solved applying the integration by parts and following two different approaches of integrating as now described.

The former approach considers the product of two functions, which are x^q and $\tilde{X}^2_{i,m}$, where the function x^q is chosen as an integrand. It will result in

$$\int_{x_i}^{x_{i+1}} x^q (\tilde{X}_{i,m})^2\,\mathrm{d}x = \left[\frac{x^{q+1}(\tilde{X}_{i,m})^2}{q+1}\right]_{x_i}^{x_{i+1}} - \frac{2}{q+1}\int_{x_i}^{x_{i+1}} x^{q+1}\tilde{X}_{i,m}\tilde{X}'_{i,m}\,\mathrm{d}x \quad (i = 1, 2, \ldots, M). \tag{B.1}$$

Substituting Eqs. (A.3) with $k = m$ in the ith integral collected on the right-hand side of Eqs. (B.1) and then applying the integration by parts, we obtain

$$\int_{x_i}^{x_{i+1}} x^{q+1}\tilde{X}_{i,m}\tilde{X}'_{i,m}\,\mathrm{d}x = -\left[\frac{x^{q+1}(\tilde{X}'_{i,m})^2}{\lambda^2_m(\alpha_1/\alpha_i)}\right]_{x_i}^{x_{i+1}} + \frac{1}{\lambda^2_m(\alpha_1/\alpha_i)} \times \int_{x_i}^{x_{i+1}} x^q \tilde{X}'_{i,m}\frac{\mathrm{d}}{\mathrm{d}x}(x\tilde{X}'_{i,m})\,\mathrm{d}x \quad (i = 1, 2, \ldots, M), \tag{B.2}$$

where the ith integral on the right-hand side of Eqs. (B.2) may be evaluated as follows. In fact, setting $x^q = x^{q-1}x$ in the derivative collected on the right-hand side of Eqs. (A.3) with $k = m$ and differentiating the product of two functions, which are x^{q-1} and $x\tilde{X}'_{i,m}$, with suitable manipulations equations (A.3) may be rewritten as

$$x^q \frac{\mathrm{d}}{\mathrm{d}x}(x\tilde{X}'_{i,m}) = -\lambda^2_m(\alpha_1/\alpha_i)x^{q+1}\tilde{X}_{i,m} - (q-1)x^q\tilde{X}'_{i,m} \quad (i = 1, 2, \ldots, M). \tag{B.3}$$

Now, substituting Eqs. (B.3) in the ith integral which appears on the right-hand side of Eqs. (B.2), we get

$$\int_{x_i}^{x_{i+1}} x^q \tilde{X}'_{i,m}\frac{\mathrm{d}}{\mathrm{d}x}(x\tilde{X}'_{i,m})\,\mathrm{d}x = -\lambda^2_m(\alpha_1/\alpha_i)\int_{x_i}^{x_{i+1}} x^{q+1}\tilde{X}_{i,m}\tilde{X}'_{i,m}\,\mathrm{d}x - (q-1)\int_{x_i}^{x_{i+1}} x^q(\tilde{X}'_{i,m})^2\,\mathrm{d}x \quad (i = 1, 2, \ldots, M). \tag{B.4}$$

In view of Eqs. (B.4) listed before, Eqs. (B.2) become

$$\int_{x_i}^{x_{i+1}} x^{q+1}\tilde{X}_{i,m}\tilde{X}'_{i,m}\,\mathrm{d}x = -\left[\frac{x^{q+1}(\tilde{X}'_{i,m})^2}{2\lambda^2_m(\alpha_1/\alpha_i)}\right]_{x_i}^{x_{i+1}} - \frac{q-1}{2\lambda^2_m(\alpha_1/\alpha_i)}\int_{x_i}^{x_{i+1}} x^q(\tilde{X}'_{i,m})^2\,\mathrm{d}x \quad (i = 1, 2, \ldots, M), \tag{B.5}$$

and then, substituting Eqs. (B.5) in Eqs. (B.1), we determine the final result in the following form:

$$\int_{x_i}^{x_{i+1}} x^q (\tilde{X}_{i,m})^2 \,\mathrm{d}x = \left[\frac{x^{q+1}(\tilde{X}_{i,m})^2}{q+1}\right]_{x_i}^{x_{i+1}} + \left[\frac{x^{q+1}(\tilde{X}'_{i,m})^2}{\lambda_m^2(\alpha_1/\alpha_i)(q+1)}\right]_{x_i}^{x_{i+1}} + \frac{q-1}{\lambda_m^2(\alpha_1/\alpha_i)(q+1)}\int_{x_i}^{x_{i+1}} x^q (\tilde{X}'_{i,m})^2 \,\mathrm{d}x \quad (i = 1, 2, \ldots, M). \tag{B.6}$$

The latter approach for solving the integrals inherent to the norm N_m considers the product of two functions, which are $x^q\tilde{X}_{i,m}$ and $\tilde{X}_{i,m}$, where the function $x^q\tilde{X}_{i,m}$ is chosen as an integrand. The reason for this choice is justified from the application of Eq. (A.3) with $k = m$, which allow the functions $x^q\tilde{X}_{i,m}$ to be easily integrated. In fact, we have

$$\int_{x_i}^{x_{i+1}} x^q (\tilde{X}_{i,m})^2 \,\mathrm{d}x = -\left[\frac{x^q\tilde{X}_{i,m}\tilde{X}'_{i,m}}{\lambda_m^2(\alpha_1/\alpha_i)}\right]_{x_i}^{x_{i+1}} + \frac{1}{\lambda_m^2(\alpha_1/\alpha_i)}\int_{x_i}^{x_{i+1}} x^q (\tilde{X}'_{i,m})^2 \,\mathrm{d}x \quad (i = 1, 2, \ldots, M). \tag{B.7}$$

Comparing Eqs. (B.6) and (B.7) yields the following result:

$$\int_{x_i}^{x_{i+1}} x^q (\tilde{X}'_{i,m})^2 \,\mathrm{d}x = \lambda_m^2(\alpha_1/\alpha_i)\left[\frac{x^{q+1}(\tilde{X}_{i,m})^2}{2}\right]_{x_i}^{x_{i+1}} + \left[\frac{x^{q+1}(\tilde{X}'_{i,m})^2}{2}\right]_{x_i}^{x_{i+1}} + (q+1)\left[\frac{x^q\tilde{X}_{i,m}\tilde{X}'_{i,m}}{2}\right]_{x_i}^{x_{i+1}} \quad (i = 1, 2, \ldots, M). \tag{B.8}$$

Substituting Eqs. (B.8) in either Eqs. (B.6) or Eqs. (B.7), we obtain the expressions (29) for the integrals inherent to the norm N_m, which are valid for composites of rectangular, cylindrical and spherical layers.

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